

Structural Models under Additional Information*

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Abstract

It has been understood that the “local” existence of the Markowitz’ optimal portfolio or the solution to the local risk minimization problem is guaranteed by some specific mathematical structures on the underlying assets price processes (called “Structure Conditions” in the literature). In this paper, we consider a semi-martingale market model (initial market model) fulfilling these structures, and an arbitrary random time that is not adapted to the flow of the “public” information. By adding additional uncertainty to the initial market model, via this random time, those structures may fail. Our aim is to address the question of how this random time will affect these structures from different perspectives. Our analysis allowed us to conclude that under some mild assumptions on the market model and the random time, these structures will remain valid on the one hand. Furthermore, we provide two examples illustrating the importance of these assumptions. On the other hand, we describe the random time models for which these structure conditions are preserved for any market model. These results are elaborated separately for the two contexts of stopping with random time and incorporating totally a specific class of random times respectively.

1 Introduction

Since the seminal work of Markowitz on the optimal portfolio, the quadratic criterion for hedging contingent claims becomes very popular and an important topic in mathematical finance, modern finance, and insurance. In this context, two main competing quadratic approaches were suggested. Precisely, the local risk minimization and the mean variance hedging. For more details about these two methods and their relationship, we refer the reader to Heath et al. [21], Cerny and Kallsen [13], Duffie and Richardson [14], Delbaen and Schachermayer [19], Biagini et al. [9], Jeanblanc et al. [28], Schweizer [33, 32], Choulli et al. [11], Laurent and Pham [30] and the references therein.

One important common feature for these methods lies in the assumptions that the market model should fulfill in order that the two methods admit solutions at least locally. These conditions are known by the “Structure Conditions” (called SC hereafter) and sound to be the alternative to non-arbitrage condition in this quadratic context. Indeed, for the case of continuous price processes, it is proved that these conditions are equivalent to the non- arbitrage of the first kind (No-Unbounded-Profit-with-Bounded-Risk, called NUPBR hereafter), or equivalent to the existence of a local martingale deflator

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for the market model. For details about these equivalence, we refer the reader to Choulli and Stricker (1996), [35]. However, in the general case, the two concepts (i.e. SC and NUPBR) differ tremendously.

Recently, there has been an upsurge interest in investigating the effect of different information levels on arbitrage theory and utility maximization problem, see [6], [12], [20], [22], [5] and the references therein. From the economic standpoint of view, information is a commodity that bears values; and economic agents desire information because it helps them to make decision and maximize their state-dependent utilities, especially when they are facing uncertainties. For more details about this economic views we refer the reader to Allen [1] and Arrow [2, 3, 4] and the references therein.

In this paper, we study the impact of some extra information/uncertainty on the Structure Conditions. This extra information comes from a random time τ that is not adapted to the public information represented by a filtration $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$. There are two mainstreams to combine the information coming from τ and \mathbb{F} : the initial enlargement and progressive enlargement of the filtration \mathbb{F} (see [29], [27], [36] and the references therein). Herein, we restrict our attention to adding the information from τ progressively to \mathbb{F} , which results in a progressive enlargement of filtration \mathbb{G} . In this paper, we are dedicated to investigate the following two questions:

For which pair (τ, S) , does S satisfy $\text{SC}(\mathbb{G})$ if S satisfies $\text{SC}(\mathbb{F})$? **(Prob1)**

and

For which τ , $\text{SC}(\mathbb{G})$ holds for any model satisfying $\text{SC}(\mathbb{F})$? **(Prob2)**

To answer the two problems **(Prob1)** and **(Prob2)**, we split the time horizon $\llbracket 0, +\infty \rrbracket$ into two disjoint intervals $\llbracket 0, \tau \rrbracket$ and $\llbracket \tau, +\infty \rrbracket$. In other words, we investigate the impact of τ on the structures of S by studying S^τ and $S - S^\tau$ separately.

This paper is organized as follows. Section 2 contains two subsections where we present the main results before τ and after τ . In Section 3, we develop the stochastic tools that would be crucial to prove the main theorems. The last section (i.e. Section 4) provides the proofs of the main theorems announced in Section 2.

2 The main results

In this section, we will summarize our main results in two subsections. The first subsection addresses the problems **(Prob1)** and **(Prob2)** under stopping with τ (i.e. we study S^τ instead), while the second subsection treats the case of τ being an honest time and focuses on $S - S^\tau$ instead. To elaborate our main results, we start by a stochastic basis $(\Omega, \mathcal{A}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where \mathbb{F} is a filtration satisfying the usual conditions of right continuity and completeness and represents the flow of “public” information over time. On this filtered probability space we consider a d -dimensional \mathbb{F} -adapted semimartingale, S , that models the tradable risky assets. We assume that S is a special semimartingale with the Doob-Meyer decomposition

$$S = S_0 + M^S + A^S, \tag{2.1}$$

where M^S is a locally square integrable \mathbb{F} -local martingale and A^S is an \mathbb{F} -predictable finite variation process. Thus, $(\Omega, \mathcal{A}, \mathbb{F}, S, \mathbb{P})$ constitutes the initial market model. In addition to this model, we consider additional information and/or uncertainty that is modelled by an \mathcal{A} -measurable random time $\tau : \Omega \rightarrow \mathbb{R}_+$ that is fixed from the beginning and for the entire paper. This random time can

represent the agent's death time, the sudden retirement time, the bankruptcy time of a firm, the default time, or any time of occurrence of an event that will affect the initial market and/or the agents. Mathematically, this random time is not a stopping time with respect to \mathbb{F} , and hence one can not confirm at time t whether this random time happened or not yet. However, using the flow of information \mathbb{F} , we can observe the survival probability of this random time. To formulate this rigorously, we associate to τ the process D and the filtration \mathbb{G} given by

$$D := I_{[\tau, +\infty[}, \quad \mathbb{G} = (\mathcal{G}_t)_{t \geq 0}, \quad \text{where } \mathcal{G}_t = \bigcap_{s > t} \left(\mathcal{F}_s \vee \sigma(D_u, u \leq s) \right). \quad (2.2)$$

The filtration \mathbb{G} is the new flow of information where the occurrence of τ can be detected. Mathematically, \mathbb{G} is the smallest right-continuous filtration that contains \mathbb{F} and makes τ being a stopping time. In the probabilistic literature, \mathbb{G} is called the progressive enlargement of \mathbb{F} with τ . In addition to \mathbb{G} and D , we associate to τ two important \mathbb{F} -supermartingales given by

$$Z_t := P(\tau > t \mid \mathcal{F}_t) \quad \text{and} \quad \tilde{Z}_t := P(\tau \geq t \mid \mathcal{F}_t). \quad (2.3)$$

The supermartingale Z (known as Azéma supermartingale) is right-continuous with left limits and coincides with the optional projection of $I_{[0, \tau]}$, while \tilde{Z} admits right limits and left limits only and is the optional projection of $I_{[0, \tau]}$. The decomposition of Z leads to another important martingale m by

$$m := Z + D^{o, \mathbb{F}}, \quad (2.4)$$

where $D^{o, \mathbb{F}}$ is the \mathbb{F} -dual optional projection of $D = I_{[\tau, \infty[}$. Furthermore, we have $\tilde{Z}_+ = Z$ and $\tilde{Z} = Z_- + \Delta m$.

Throughout the paper, the filtration \mathbb{H} denotes an arbitrary filtration satisfying the usual conditions. As usual, the set of \mathbb{H} -martingales that are p -integrable ($p \geq 1$) will be denoted by $\mathcal{M}^p(\mathbb{H})$ and $\mathcal{A}^+(\mathbb{H})$ denotes the set of increasing, right-continuous, \mathbb{H} -adapted and integrable processes. If $\mathcal{C}(\mathbb{H})$ is a class of processes for the filtration \mathbb{H} , we denote by $\mathcal{C}_0(\mathbb{H})$ the set of processes $X \in \mathcal{C}(\mathbb{H})$ with $X_0 = 0$, and by \mathcal{C}_{loc} the set of processes X such that there exists a sequence of \mathbb{H} -stopping times, $(T_n)_{n \geq 1}$, that increases to $+\infty$ and the stopped process X^{T_n} belongs to $\mathcal{C}(\mathbb{H})$. We put $\mathcal{C}_{0, loc} = \mathcal{C}_0 \cap \mathcal{C}_{loc}$.

From time to time, we need to calculate predictable projection and compensator under different filtrations. To distinguish the effect, we shall use ${}^{p, \mathbb{H}}(V)$ and $(V)^{p, \mathbb{H}}$ to specify the filtration \mathbb{H} . When the quadratic variation process $[\cdot, \cdot]$ is \mathbb{H} -locally integrable, we denote $\langle \cdot, \cdot \rangle^{p, \mathbb{H}}$ as its compensator. For any \mathbb{H} -locally square integrable local martingale X , we denote $L_{loc}^2(X, \mathbb{H})$ the set of predictable processes θ that are (X, \mathbb{H}) -integrable (in the semimartingale sense) and the resulting integral $\theta \cdot X$ is an \mathbb{H} -locally square integrable local martingale.

Below, we recall the notion of Structure Conditions that we will address in this paper. It goes back to Schweizer [33].

Definitions 2.1. *Let X be an \mathbb{H} -adapted process. We say that X satisfies the Structure Conditions under \mathbb{H} (hereafter, $SC(\mathbb{H})$), if there exist $M \in \mathcal{M}_{0, loc}^2(P, \mathbb{H})$ and $\lambda \in L_{loc}^2(M, \mathbb{H})$ such that*

$$X = X_0 + M - \lambda \cdot \langle M, M \rangle^{\mathbb{H}}. \quad (2.5)$$

For more details about Structure Conditions and other related properties, we refer the reader to Schweizer [33, 32], Choulli and Stricker [10], and the references therein.

Below, we prove a simple but useful lemma for Structure Conditions.

Lemma 2.2. *Let V be an \mathbb{H} -predictable with finite variation process. Then, V satisfies $SC(\mathbb{H})$ if and only if V is constant (i.e. $V_t \equiv V_0$, $t \geq 0$).*

Proof. If V satisfies $SC(\mathbb{H})$, then there exist an \mathbb{H} -local martingale M^V and an \mathbb{H} -predictable process $\lambda^{\mathbb{H}} \in L_{loc}^2(M)$ such that $V = V_0 + M^V + \lambda^{\mathbb{H}} \cdot \langle M^V, M^V \rangle^{\mathbb{H}}$. Therefore, M^V is an \mathbb{H} -predictable local martingale with finite variation. Hence M is null, and $V \equiv V_0$. This ends the proof of the lemma. \square

The following lemma explains why one can split the study of the Structure Conditions of (S, \mathbb{G}) into two separate cases, namely (S^τ, \mathbb{G}) and $(S - S^\tau, \mathbb{G})$.

Lemma 2.3. *(S, \mathbb{G}) satisfies the Structure Conditions if and only if both (S^τ, \mathbb{G}) and $(S - S^\tau, \mathbb{G})$ do.*

Proof. The proof follows immediately from the definition. \square

2.1 Structure Conditions under Stopping with Random Time

In this section, we will investigate and quantify the effect of stopping with τ on the Structure Conditions in two different ways. Below, we state the first main result of this subsection, which provides sufficient condition on τ and S for which the Structure Conditions are preserved after stopping with τ . This answers partially the problem **(Prob1)**.

Theorem 2.4. *Consider any random time τ and suppose that S satisfies $SC(\mathbb{F})$ with the Doob-Meyer decomposition $S := S_0 + M^S + A^S$. If*

$$\{\Delta M^S \neq 0\} \cap \{\tilde{Z} = 0\} \cap \{Z_- > 0\} = \emptyset, \quad (2.6)$$

then S^τ satisfies $SC(\mathbb{G})$.

Proof. The proof requires many intermediary results that are interesting in themselves. Therefore, this proof is delegated to Section 4. \square

Our second main theorem of this subsection answers completely the problem **(Prob2)**, and describes the random time models for which the Structure Conditions are preserved after stopping with τ .

Theorem 2.5. *Let τ be a random time. Then, the following assertions are equivalent.*

- (a) *The thin set $\{\tilde{Z} = 0\} \cap \{Z_- > 0\}$ is evanescent.*
- (b) *For any process X satisfying $SC(\mathbb{F})$, X^τ satisfies $SC(\mathbb{G})$.*

Proof. The proof of (a) \Rightarrow (b) follows immediately from Theorem 2.4. To prove the reverse sense, we assume that assertion (b) holds. Remark that $\{\tilde{Z} = 0\} \cap \{Z_- > 0\} \subset \{\Delta m \neq 0\}$, it is a thin set, and consider T a stopping time such that $\llbracket T \rrbracket \subset \{\tilde{Z} = 0\} \cap \{Z_- > 0\}$. Then, M^τ satisfies $SC(\mathbb{G})$, where

$$M = V - \tilde{V} \in \mathcal{M}_0(\mathbb{F}), \quad V := I_{\llbracket T, +\infty \rrbracket} \quad \text{and} \quad \tilde{V} := (V)^{p, \mathbb{F}}. \quad (2.7)$$

Since $\tau < T$, P -a.s. on $\{T < +\infty\}$ (due to $\tilde{Z}_T = 0$ on $\{T < +\infty\}$), we deduce that

$$M^\tau = -(\tilde{V})^\tau \text{ is } \mathbb{G}\text{-predictable and satisfies } SC(\mathbb{G}). \quad (2.8)$$

Hence, by combining (2.8) and Lemma 2.2, we conclude that M^τ is null (i.e. $(\tilde{V})^\tau = 0$), or equivalently

$$0 = E(\tilde{V}_\tau) = E\left(\int_0^{+\infty} Z_{s-} d\tilde{V}_s\right) = E(Z_{T-} I_{\{T < +\infty\}}). \quad (2.9)$$

Thus $T = +\infty$, P -a.s., and the thin set $\{\tilde{Z} = 0\} \cap \{Z_- > 0\}$ is evanescent (see Proposition 2.18 on Page 20 in [26]). This ends the proof of the theorem. \square

Corollary 2.6. *For any random time τ , if either m is continuous or Z is positive, then S^τ satisfies $SC(\mathbb{G})$ for any process S that satisfies $SC(\mathbb{F})$.*

Proof. Under the condition either m is continuous or Z is positive, the set $\{\tilde{Z} = 0\} \cap \{Z_- > 0\}$ is evanescent. Hence, the proof of the corollary follows immediately from Theorem 2.5. Below, we detail a direct proof for the case when m is continuous. In fact this direct proof contains the key ideas for the proof of Theorem 2.4, but with more dedicate arguments due to the jumps.

Suppose that S satisfies $SC(\mathbb{F})$ with the canonical decomposition $S = S_0 + M - \lambda \cdot \langle M, M \rangle^\mathbb{F}$, where $M \in \mathcal{M}_{loc}^2(\mathbb{F})$ and $\lambda \in L_{loc}^2(M, \mathbb{F})$. Put the \mathbb{G} -locally square integrable local martingale (see Jeulin [29])

$$\widehat{M} := I_{[0, \tau]} \cdot M - (Z_-)^{-1} I_{[0, \tau]} \cdot \langle M, m \rangle^\mathbb{F}. \quad (2.10)$$

Then, the canonical decomposition of S^τ under \mathbb{G} has the form of

$$S^\tau = S_0 + \widehat{M} - \lambda I_{[0, \tau]} \cdot \langle M, M \rangle^\mathbb{F} + (Z_-)^{-1} I_{[0, \tau]} \cdot \langle M, m \rangle^\mathbb{F}. \quad (2.11)$$

Thus, the proof will follow as long as we find a \mathbb{G} -predictable process $\widehat{\lambda} \in L_{loc}^2(\widehat{M}, \mathbb{G})$ such that

$$-\lambda I_{[0, \tau]} \cdot \langle M, M \rangle^\mathbb{F} + (Z_-)^{-1} I_{[0, \tau]} \cdot \langle M, m \rangle^\mathbb{F} = \widehat{\lambda} \cdot \widehat{M}^\mathbb{G}. \quad (2.12)$$

Indeed, since m is continuous, the Galtchouk-Kunita-Watanabe decomposition of m with respect to M under \mathbb{F} implies the existence of an \mathbb{F} -predictable process $\beta_m \in L_{loc}^2(M)$ and a locally square integrable \mathbb{F} -local martingale M^\perp such that

$$m = m_0 + \beta_m \cdot M + M^\perp \quad \text{and} \quad \langle M, M^\perp \rangle^\mathbb{F} = 0. \quad (2.13)$$

Therefore,

$$\begin{aligned} -\lambda I_{[0, \tau]} \cdot \langle M, M \rangle^\mathbb{F} + \frac{1}{Z_-} I_{[0, \tau]} \cdot \langle M, m \rangle^\mathbb{F} &= -\lambda I_{[0, \tau]} \cdot \langle M, M \rangle^\mathbb{F} + \frac{\beta_m}{Z_-} I_{[0, \tau]} \cdot \langle M, M \rangle^\mathbb{F} \\ &= \left(-\lambda + \frac{\beta_m}{Z_-} \right) I_{[0, \tau]} \cdot \langle M, M \rangle^\mathbb{F}. \end{aligned} \quad (2.14)$$

It is easy to prove that $I_{[0, \tau]} \cdot \langle M \rangle^\mathbb{F} = I_{[0, \tau]} \cdot \langle M \rangle^\mathbb{G} = I_{[0, \tau]} \cdot \langle \widehat{M} \rangle^\mathbb{G}$, due to the continuity of m . Thus, we obtain

$$S^\tau = S_0 + \widehat{M} - \widehat{\lambda} \cdot \widehat{M}^\mathbb{G}, \quad (2.15)$$

where $\widehat{\lambda} := \left(\lambda - \frac{\beta_m}{Z_-} \right) I_{[0, \tau]}$. It is obvious that $\widehat{\lambda} \in L_{loc}^2(\widehat{M})$ due to the local boundedness of $(Z_-)^{-1} I_{[0, \tau]}$. This ends the proof of the Corollary. \square

One may wonder what could happen when the condition (2.6) fails. Below, we provide an example when $\{\tilde{Z} = 0 < Z_-\}$ is nonempty, and S^τ fails to satisfy $SC(\mathbb{G})$ (for the arbitrage opportunities in the example, we refer to Aksamit et al. [6]).

Proposition 2.7. *Suppose that the stochastic basis $(\Omega, \mathbb{A}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)$ supports a Poisson process N with intensity λ , and the stock price –denoted by X – is given by*

$$dX_t = X_{t-} \psi dM_t, \quad \text{where, } \psi > -1, \text{ and } \psi \neq 0, \quad M_t = N_t - \lambda t.$$

If

$$\tau = k_1 T_1 + k_2 T_2, \quad \text{where } T_i = \inf\{t \geq 0 : N_t \geq i\}, \quad i \geq 1, \quad k_1, k_2 > 0, \text{ and } k_1 + k_2 = 1,$$

then X^τ does not satisfy $SC(\mathbb{G})$.

Proof. We recall from Aksamit et al. [6] that the Azéma supermartingale Z and m take the forms of:

$$Z = I_{\llbracket 0, T_1 \rrbracket} + \phi^m I_{\llbracket T_1, T_2 \rrbracket}, \quad m = 1 - \phi^m I_{\llbracket T_1, T_2 \rrbracket} \cdot M, \quad \text{where } \phi_t^m = e^{-\lambda \frac{k_1}{k_2}(t-T_1)}. \quad (2.16)$$

Then, it is easy to calculate that

$$\begin{aligned} \frac{1}{Z_-} I_{\llbracket 0, \tau \rrbracket} \cdot \langle X, m \rangle_t &= \frac{-1}{Z_-} I_{\llbracket T_1, \tau \rrbracket} X_- \psi \phi^m \cdot \langle M \rangle_t \\ &= -\lambda \int_0^t \frac{1}{Z_{u-}} I_{\llbracket T_1, \tau \rrbracket} X_{u-} \psi_u \phi_u^m du = -\lambda \int_0^t X_{u-} \psi_u I_{\llbracket T_1, \tau \rrbracket} du \end{aligned}$$

and

$$\begin{aligned} \langle \hat{X}, \hat{X} \rangle_t^{\mathbb{G}} &= I_{\llbracket 0, \tau \rrbracket} \cdot \langle X, X \rangle_t + \frac{1}{Z_-} I_{\llbracket 0, \tau \rrbracket} \cdot \left(\sum \Delta m (\Delta X)^2 \right)_t^{p, \mathbb{F}} \\ &= \lambda \int_0^t X_{u-}^2 \psi_u^2 I_{\llbracket 0, \tau \rrbracket} du - \lambda \int_0^t \frac{1}{Z_{u-}} X_{u-}^2 \psi_u^2 \phi_u^m I_{\llbracket T_1, \tau \rrbracket} du \\ &= \lambda \int_0^t X_{u-}^2 \psi_u^2 I_{\llbracket 0, T_1 \rrbracket} du, \end{aligned}$$

where \hat{X} is defined via (3.29). Hence, there is no \mathbb{G} -predictable process $\hat{\lambda} \in L_{loc}^2(\hat{X})$ satisfying

$$\frac{1}{Z_-} I_{\llbracket 0, \tau \rrbracket} \cdot \langle X, m \rangle = \hat{\lambda} \cdot \langle \hat{X}, \hat{X} \rangle^{\mathbb{G}}, \quad (2.17)$$

since $\llbracket T_1, \tau \rrbracket$ and $\llbracket 0, T_1 \rrbracket$ are disjoint. This ends the proof of the proposition. \square

2.2 Structure Conditions under a Class of Honest Times

In this section, we focus on answering the two problems **(Prob1)** and **(Prob2)** when we totally incorporate a random time. This can be achieved by splitting the whole half line into two stochastic intervals $\llbracket 0, \tau \rrbracket$ and $\llbracket \tau, +\infty \rrbracket$. The first part, i.e. S^τ is already studied in the previous section. Thus, this section will concentrate on studying the Structure Conditions of S on the stochastic interval $\llbracket \tau, +\infty \rrbracket$. The first obstacle that one can face in this study is how far the (H') -hypothesis is preserved on this interval (i.e. any \mathbb{F} -semimartingale stays a \mathbb{G} -semimartingale)? To overcome this difficulty that is not our main focus in the paper, we restrict our study to the important class of random times, called honest times. Below, we recall their definition.

Definitions 2.8. A random time τ is called an honest time, if for any t , there exists an \mathcal{F}_t -measurable random variable τ_t such that $\tau I_{\{\tau < t\}} = \tau_t I_{\{\tau < t\}}$.

We refer to Jeulin [29, Chapter 4] for more information on honest times. Throughout this section, the random time τ is supposed to be honest and satisfies

$$Z_\tau < 1, \quad P - a.s. \quad (2.18)$$

Remark 2.9. This assumption is also crucial for the validity of No-Unbounded-Profit-with-Bounded-Risk after an honest time. We refer to Choulli et al [12] for more details on this subject.

Now, we state the two main theorems in this section. This answers partially the problem **(Prob1)** and completely the problem **(Prob2)**.

Theorem 2.10. *Let τ be an honest time satisfying $Z_\tau < 1, a.s..$ If S is a process satisfying $SC(\mathbb{F})$ and*

$$\{\Delta M^S \neq 0\} \cap \{\tilde{Z} = 1 > Z_-\} = \emptyset, \quad (2.19)$$

then $S - S^\tau$ satisfies $SC(\mathbb{G})$.

Proof. The proof would be detailed in Section 4. □

Theorem 2.11. *Let τ be an \mathbb{F} -honest time satisfying $Z_\tau < 1, a.s..$ Then, the following are equivalent.*

- (a) *The thin set $\{\tilde{Z} = 1 > Z_-\}$ is evanescent.*
- (b) *For any process X satisfying $SC(\mathbb{F})$, $X - X^\tau$ satisfies $SC(\mathbb{G})$.*

Proof. The proof of (a) \Rightarrow (b) is a direct consequence of Theorem 2.10. To prove the reverse, we assume that assertion (b) holds, and follow similar steps as in the proof of Theorem 2.5. For the sake of completeness, we give the full details. Since $\{\tilde{Z} = 1\} \cap \{Z_- < 1\} \subset \{\Delta m \neq 0\}$, it is a thin set. Let T be any stopping time such that $\llbracket T \rrbracket \subset \{\tilde{Z} = 1\} \cap \{Z_- < 1\}$. Then, consider the following \mathbb{F} -martingale,

$$M = V - \tilde{V} \in \mathcal{M}_0(\mathbb{F}), \text{ where } V := I_{\llbracket T, +\infty \rrbracket} \text{ and } \tilde{V} := (V)^{p, \mathbb{F}}. \quad (2.20)$$

Since $\{t > \tau\} \subset \{\tilde{Z}_t < 1\}$ (see Jeulin [29] or Choulli et al. [12]) and $\tilde{Z}_T = 1$, we deduce that $\tau \geq T$, $P - a.s.$, and

$$M - M^\tau = -I_{\llbracket \tau, +\infty \rrbracket} \cdot \tilde{V} \quad (2.21)$$

which is \mathbb{G} -predictable and satisfies $SC(\mathbb{G})$. By combining (2.21) with Lemma 2.2, we conclude that $M - M^\tau$ is null (i.e. $I_{\llbracket \tau, +\infty \rrbracket} \cdot \tilde{V} = 0$). Thus, we get

$$0 = E \left(I_{\llbracket \tau, +\infty \rrbracket} \cdot \tilde{V}_\infty \right) = E \left(\int_0^{+\infty} (1 - Z_{s-}) d\tilde{V}_s \right) = E \left((1 - Z_{T-}) I_{\{T < +\infty\}} \right), \quad (2.22)$$

or equivalently $(1 - Z_{T-}) I_{\{T < +\infty\}} = 0$ that implies that $T = +\infty$, $P - a.s..$ Therefore the thin set $\{\tilde{Z} = 1\} \cap \{Z_- < 1\}$ is evanescent (see Proposition 2.18 on Page 20 in [26]). This ends the proof of the theorem. □

As a simple corollary, we have

Corollary 2.12. *For any \mathbb{F} -honest time τ satisfying $Z_\tau < 1, P - a.s.$, if m is continuous, then $S - S^\tau$ satisfies $SC(\mathbb{G})$ for any process S that satisfies $SC(\mathbb{F})$.*

Proof. It is enough to notice that under the condition m is continuous, the set $\{1 = \tilde{Z} > Z_-\} \subset \{\Delta m \neq 0\}$ is empty. □

Example 2.13. *Herein, we present an example that when $\{\Delta X \neq 0\} \cap \{1 = \tilde{Z} > Z_-\} \neq \emptyset$, $X - X^\tau$ fails to satisfy $SC(\mathbb{G})$. We suppose given a Poisson process N , with intensity rate $\lambda > 0$, and the natural filtration \mathbb{F} . The stock price process is given by*

$$dX_t = X_{t-} \sigma dM_t, \quad X_0 = 1, \quad M_t := N_t - \lambda t,$$

or equivalently $X_t = \exp(-\lambda \sigma t + \ln(1 + \sigma) N_t)$, where $\sigma > 0$. In what follows, we introduce the notations

$$a := -\frac{1}{\ln(1 + \sigma)} \ln b, \quad 0 < b < 1, \quad \mu := \frac{\lambda \sigma}{\ln(1 + \sigma)} \quad \text{and} \quad Y_t := \mu t - N_t.$$

We associate to the process Y its ruin probability, denoted by $\Psi(x)$ given by, for $x \geq 0$,

$$\Psi(x) = P(T^x < \infty), \quad \text{with} \quad T^x = \inf\{t : x + Y_t < 0\}. \quad (2.23)$$

Proposition 2.14. *Consider the model and its notations in Example 2.13, and the following random time*

$$\tau := \sup\{t : X_t \geq b\} = \sup\{t : Y_t \leq a\}. \quad (2.24)$$

Then $X - X^\tau$ fails to satisfy $SC(\mathbb{G})$.

Proof. We recall from Aksamit et al. [6] that the supermartingale Z and m are given by

$$\begin{aligned} Z_t &= P(\tau > t | \mathcal{F}_t) = \Psi(Y_t - a)I_{\{Y_t \geq a\}} + I_{\{Y_t < a\}} = 1 + I_{\{Y_t \geq a\}}(\Psi(Y_t - a) - 1), \\ \Delta m &= I_{\{Y_- > a+1\}}(\Psi(Y_- - a - 1) - 1)\Delta N - I_{\{Y_- > a\}}(\Psi(Y_- - a) - 1)\Delta N \\ &:= I_{\{Y_- > a+1\}}\phi_1\Delta N - I_{\{Y_- > a\}}\phi_2\Delta N, \end{aligned}$$

where Ψ is defined in (2.23). Then it is easy to calculate that

$$\begin{aligned} \frac{1}{1 - Z_-} I_{\tau, +\infty} \cdot \langle X, m \rangle_t &= -\lambda\sigma \int_0^t \frac{X_{u-}}{\phi_2(u)} \{I_{\{Y_{u-} > a+1\}}\phi_1(u) - I_{\{Y_{u-} > a\}}\phi_2(u)\} I_{\tau, +\infty} du, \text{ and} \\ I_{\tau, +\infty} \cdot \left\langle \widehat{X}, \widehat{X} \right\rangle_t^{\mathbb{G}} &= I_{\tau, +\infty} \frac{1}{1 - Z_-} \cdot \left((1 - \widetilde{Z}) \cdot [X, X] \right)_t^{p, \mathbb{F}} \\ &= \lambda\sigma^2 \int_0^t \frac{\phi_1(u)X_{u-}^2}{\phi_2(u)} I_{\{Y_{u-} > a+1\}} I_{\tau, +\infty} du, \end{aligned}$$

where \widehat{X} is defined via (3.50). Notice that on the interval $\{a + 1 \geq Y_- > a\}$,

$$\frac{1}{1 - Z_-} I_{\tau, +\infty} \cdot \langle X, m \rangle_t = \lambda\sigma \int_0^t X_{u-} I_{\{Y_{u-} > a\}} I_{\tau, +\infty} du, \text{ while } I_{\tau, +\infty} \cdot \left\langle \widehat{X}, \widehat{X} \right\rangle^{\mathbb{G}} = 0.$$

Hence, there is no \mathbb{G} -predictable process $\widehat{\lambda} \in L_{loc}^2(X)$ such that

$$\frac{1}{1 - Z_-} I_{\tau, +\infty} \cdot \langle X, m \rangle = I_{\tau, +\infty} \widehat{\lambda} \cdot \left\langle \widehat{X}, \widehat{X} \right\rangle^{\mathbb{G}}.$$

Hence, $X - X^\tau$ fails to satisfy $SC(\mathbb{G})$. □

3 The Key Stochastic Tools

Here, we will provide the crucial stochastic tools for the proof of two main theorems announced in Section 2. This section contains three subsections. In subsection 3.1, we recall a Lazaro and Yor's result that we extend to the case of locally square integrable martingales. Then, we give the definition and important properties of the optional stochastic integral. In subsections 3.2-3.3, we provide innovative lemmas and propositions that play key roles in the proof of the two main theorems.

3.1 A Lazaro–Yor's Representation and Optional Stochastic Integral

This subsection introduces and slightly extends two stochastic tools that are pillars in our analysis, namely the Lazaro-Yor's representation and the optional stochastic integral. The following extends the representation of Lazaro and Yor [31] to the “local and dynamic” framework.

Lemma 3.1. *Let M be a local martingale and $(Y_t^n), (Y_t)$ be two uniform integrable martingales such that Y_∞^n converges to Y_∞ weakly in L^1 . If Y^n admits representation as stochastic integrals with respect to M , i.e.*

$$Y_t^n = \int_0^t \phi_s^n dM_s. \quad (3.25)$$

Then there exists a predictable process ϕ such that $Y_t = \int_0^t \phi_s dM_s$.

Proof. For the proof we refer the reader to Lazaro and Yor [31] or Jacod [25]. \square

To extend this lemma to the dynamic case, we first define the weak convergence in the space $\mathcal{M}_{loc}^2(\mathbb{H})$.

Definitions 3.2. A sequence of elements of $\mathcal{M}_{loc}^2(\mathbb{H})$, $(Y^n)_{n \geq 1}$, is said to converge weakly in $\mathcal{M}_{loc}^2(\mathbb{H})$ if there exist $Y \in \mathcal{M}_{loc}^2(\mathbb{H})$, and a sequence of \mathbb{H} -stopping times that increases to infinity, $(\sigma_k)_{k \geq 1}$ such that for each $k \geq 1$, the sequence $(Y_{\sigma_k}^n)_{n \geq 1}$ converges weakly to Y_{σ_k} in $L^2(\mathbb{P})$.

Below, we extend Lazaro-Yor's lemma to the dynamic and local framework, which will play important rôles in the proofs of the main results.

Lemma 3.3. Let M be a locally square integrable local martingale, and $(\phi_n)_{n \geq 1}$ be a sequence of predictable processes that belong to $L_{loc}^2(M)$. If $(\phi_n \cdot M)$ converges weakly in $\mathcal{M}_{loc}^2(\mathbb{H})$, then there exists $\phi \in L_{loc}^2(M)$ such that $\phi \cdot M$ coincides with its limit.

Proof. If $(\phi_n \cdot M)$ converges weakly in $\mathcal{M}_{loc}^2(\mathbb{H})$ with the localizing sequence $(\sigma_k)_{k \geq 1}$, then there exists $Y \in \mathcal{M}_{loc}^2$ such that Y^{σ_k} and $(\phi_n \cdot M)^{\sigma_k}$ are square integrable martingales and $(\phi_n \cdot M)_{\sigma_k}$ converges weakly in $L^1(P)$ to Y_{σ_k} . Hence, a direct application of Lemma 3.1 implies the existence of a predictable process ψ^k that is M^{σ_k} -integrable and the resulting integrable $(\psi^k \cdot M^{\sigma_k})$ coincides with Y^{σ_k} (due to the uniqueness of the limit). Then, by putting

$$\phi := \sum_{k=1}^{+\infty} \psi^k I_{\llbracket \sigma_{k-1}, \sigma_k \rrbracket},$$

we can easily deduce that $\phi \in L_{loc}^2(M)$, as well as $Y = \phi \cdot M$. This ends the proof of the corollary. \square

The second stochastic notion (that is essential in our analysis) is the optional stochastic integral (or compensated stochastic integral), which is defined in [25] (Chapter III.4.b p. 106-109) and also studied in [18] (Chapter VIII.2 sections 32-35 p. 356-361). It is a generalization of predictable stochastic integral with respect to local martingale, that appears naturally in this context of Structure Conditions under enlargement of filtration.

Definitions 3.4. [[25] Definition (3.80)] Let N be an \mathbb{H} -local martingale with continuous martingale part N^c , H an \mathbb{H} -optional process, and $p \in [1, +\infty)$.

(a) The process H is said to be p -integrable with respect to N if ${}^{p, \mathbb{H}}H$ is N^c integrable, ${}^p(|H \Delta N|) < +\infty$, and the process

$$\left(\sum_{s \leq t} \left(H \Delta N - {}^{p, \mathbb{H}}(H \Delta N) \right)^2 \right)^{p/2} \quad (3.26)$$

is locally integrable. The set of p -integrable processes with respect to N is denoted by ${}^o L_{loc}^p(N, \mathbb{H})$.

(b) For $H \in {}^o L_{loc}^p(N, \mathbb{H})$, the compensated stochastic integral of H with respect to N will be denoted by, $H \odot N$, that is the unique p -locally integrable local martingale M such that

$$M^c = {}^{p, \mathbb{H}}H \cdot N^c \quad \text{and} \quad \Delta M = H \Delta N - {}^{p, \mathbb{H}}(H \Delta N). \quad (3.27)$$

We recall some important properties of the optional stochastic integral:

Proposition 3.5. [18] (a) *The optional stochastic integral $M = H \odot N$ is the unique \mathbb{H} -local martingale such that, for any \mathbb{H} -local martingale Y ,*

$$\mathbb{E}([M, Y]_\infty) = \mathbb{E}\left(\int_0^\infty H_s d[N, Y]_s\right).$$

(b) *The process $[H \odot N, Y] - H \cdot [N, Y]$ is an \mathbb{H} -local martingale. Therefore, $[M, Y] \in \mathcal{A}_{loc}(\mathbb{H})$ if and only if $H \cdot [N, Y] \in \mathcal{A}_{loc}(\mathbb{H})$ and in this case we have*

$$\langle H \odot N, Y \rangle^{\mathbb{H}} = (H \cdot [N, Y])^{p, \mathbb{H}}.$$

Lemma 3.6. *Let H be an \mathbb{H} -optional process and N be an \mathbb{H} -local martingale. Then $H \in {}^o L_{loc}^p(N, \mathbb{H})$ and $H^2 \cdot [N, N]$ has finite values if and only if $^p(|H \Delta N|) < +\infty$, and $(H^2 \cdot [N, N])^{p/2} \in \mathcal{A}_{loc}^+(\mathbb{H})$.*

Proof. The proof of the lemma can be found in He et al. [23] (see Theorem 9.10 and the Remarks on pages 232-233). \square

We end this section with a lemma that calculates the sharp bracket of two optional stochastic integrals.

Lemma 3.7. *Consider a filtration \mathbb{H} satisfying the usual conditions. Let M and N be two \mathbb{H} -locally square integrable local martingales such that $H \in {}^o L_{loc}^2(N, \mathbb{H})$ and $K \in {}^o L_{loc}^2(M, \mathbb{H})$. Then, we have*

$$\langle H \odot N, K \odot M \rangle^{\mathbb{H}} = \left(HK \cdot [M, N] \right)^{p, \mathbb{H}} - \sum {}^{p, \mathbb{H}}(H \Delta N) {}^{p, \mathbb{H}}(K \Delta M). \quad (3.28)$$

Proof. An immediate application of Proposition 3.5(b) leads to the existence of an \mathbb{H} -local martingale L such that

$$[H \odot N, K \odot M] = L + H \cdot [N, K \odot M] =: B$$

Using Proposition 3.5(b) again, we obtain that

$$\begin{aligned} B &= L + H \cdot [N^c, (K \odot M)^c] + \sum H \Delta N \left(K \Delta M - {}^{p, \mathbb{H}}(K \Delta M) \right) \\ &= L + H \cdot [N^c, (K \odot M)^c] + \sum HK \Delta N \Delta M - \sum H \Delta N {}^{p, \mathbb{H}}(K \Delta M) \\ &= L + HK \cdot [N^c, M^c] + HK \cdot [M, N]^d \\ &\quad - \sum \left(H \Delta N - {}^{p, \mathbb{H}}(H \Delta N) \right) {}^{p, \mathbb{H}}(K \Delta M) - \sum {}^{p, \mathbb{H}}(H \Delta N) {}^{p, \mathbb{H}}(K \Delta M). \end{aligned}$$

It is easy to see that ${}^{p, \mathbb{H}}(K \Delta M)$ is locally bounded and due to (3.27), the process

$$\sum \left(H \Delta N - {}^{p, \mathbb{H}}(H \Delta N) \right) {}^{p, \mathbb{H}}(K \Delta M)$$

is a local martingale. Therefore, we get

$$\langle H \odot N, K \odot M \rangle^{\mathbb{H}} = \left(HK \cdot [M, N] \right)^{p, \mathbb{H}} - \sum {}^{p, \mathbb{H}}(H \Delta N) {}^{p, \mathbb{H}}(K \Delta M).$$

This ends the proof of the lemma. \square

3.2 The key stochastic results for the part up to random horizon

In this section, we will present some crucial lemmas and propositions that we need to prove the main theorem 2.4. First, let us point out that to prove $S^\tau \in \text{SC}(\mathbb{G})$, it is essential to find a \mathbb{G} -predictable process $\phi^\mathbb{G} \in L_{loc}^2(\widehat{M}^S)$ such that $I_{[0,\tau]} \cdot \langle m, M^S \rangle = \phi^\mathbb{G} \cdot \langle \widehat{M}^S \rangle^\mathbb{G}$. As one could predict, the difficulty lies in the existence and the locally square integrability of $\phi^\mathbb{G}$. Due to Jeulin [29], the H' hypothesis is preserved, i.e. any \mathbb{F} -semimartingale stays a \mathbb{G} -semimartingale on $[0, \tau]$.

Lemma 3.8. *To any \mathbb{F} -local martingale M , we associate to it the process $\widehat{M}^{(b)}$ given by*

$$\widehat{M}_t^{(b)} := M_{t \wedge \tau} - \int_0^{t \wedge \tau} \frac{d\langle M, m \rangle_s^\mathbb{F}}{Z_{s-}}, \quad (3.29)$$

which is a \mathbb{G} -local martingale.

Proof. The proof of this lemma can be found in Jeulin [29]. □

On the stochastic integral $[0, \tau]$, it is worthy to keep in mind that for any \mathbb{F} -local martingale M , $\widehat{M}^{(b)}$ would be defined via (3.29) in what follows.

Below, we recall an important lemma due to Choulli et al. [12].

Lemma 3.9 ([12]). *The following assertions hold.*

(a) *For any \mathbb{F} -adapted process V with locally integrable variation, we have*

$$(V^\tau)^{p,\mathbb{G}} = (Z_-)^{-1} I_{[0,\tau]} \cdot (\widetilde{Z} \cdot V)^{p,\mathbb{F}}. \quad (3.30)$$

(b) *For any \mathbb{F} -local martingale M , we have, on $[0, \tau]$*

$$p,\mathbb{G} \left(\frac{\Delta M}{\widetilde{Z}} \right) = \frac{p,\mathbb{F} \left(\Delta M I_{\{\widetilde{Z} > 0\}} \right)}{Z_-}, \quad \text{and} \quad p,\mathbb{G} \left(\frac{1}{\widetilde{Z}} \right) = \frac{p,\mathbb{F} \left(I_{\{\widetilde{Z} > 0\}} \right)}{Z_-}. \quad (3.31)$$

Remark 3.10. *To explain the main difficulty that one will encounter when proving the Structure Conditions for S^τ , we assume that S is an \mathbb{F} -local martingale. Then, due to Choulli et al. [12], the condition $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0\} \cap \{Z_- > 0\} = \emptyset$ implies that*

$$\mathcal{E} \left(\frac{-Z_-^2}{Z_-^2 + \Delta \langle m \rangle} \frac{1}{\widetilde{Z}} I_{[0,\tau]} \odot \widehat{m}^{(b)} \right) \quad (3.32)$$

is a local martingale density for S^τ . However, because of the term $1/\widetilde{Z}$, in general it is not locally square integrable. To overcome this difficulty, one applies the Galtchouk-Kunita-Watanabe decomposition of

$$H^n \odot \widehat{m}^{(b)} := \frac{1}{\widetilde{Z}} I_{\{\widetilde{Z} \geq 1/n\}} I_{[0,\tau]} \odot \widehat{m}^{(b)}$$

with respect to $\widehat{M}^{(b)}$, and obtain

$$H^n \odot \widehat{m}^{(b)} = \phi^n \cdot \widehat{M}^{(b)} + L^n, \quad \text{where } \phi^n \in L_{loc}^2(\widehat{M}^{(b)}), \text{ and } L^n \perp \widehat{M}^{(b)}.$$

Then, the main difficulty lies in proving the weak convergence of $\phi^n \cdot \widehat{M}^{(b)}$ and use Lemma 3.1 and Lemma 3.3 afterwards to conclude that $\phi^n \cdot \widehat{M}^{(b)}$ converge weakly in $\mathcal{M}_{loc}^2(\mathbb{G})$ to a locally square integrable local martingale having the form of $\Phi_1 \cdot \widehat{M}^{(b)}$. Therefore, this will establish the connection between $\langle \widehat{M}^{(b)} \rangle^\mathbb{G}$ and $\langle m, M \rangle^\mathbb{F}$ on the stochastic interval $[0, \tau]$.

We start with a proposition that is dealing with weakly convergence in \mathcal{M}_{loc}^2 that would be frequently used in what follows.

Proposition 3.11. *Let M, N be two \mathbb{H} -locally square integrable local martingales and H be an non-negative \mathbb{H} -optional process such that $H \cdot [M, M]$ and $H \cdot [N, N]$ have finite values. If $H^n := HI_{\{H \leq n\}}$, then the following assertions hold.*

- (a) *If $\sqrt{H} \in {}^oL_{loc}^2(M)$, then $\sqrt{H}I_{\{H \leq n\}} \odot M$ converges in $\mathcal{M}_{loc}^2(\mathbb{H})$ to $\sqrt{H} \odot M$.*
- (b) *If $\sqrt{H} \in {}^oL_{loc}^2(M) \cap {}^oL_{loc}^2(N)$, then there exists a sequence of \mathbb{H} -stopping times $(\eta_k)_{k \geq 1}$ increasing to infinity such that for all $k \geq 1$,*

$$\left\langle \sqrt{H}I_{\{H \leq n\}} \odot M, \sqrt{H}I_{\{H \leq n\}} \odot N \right\rangle_{\eta_k} \text{ converges in } L^1 \text{ to } \left\langle \sqrt{H} \odot M, \sqrt{H} \odot N \right\rangle_{\eta_k}.$$

- (c) *Suppose that $\sqrt{H} \in {}^oL_{loc}^2(Y) \cap {}^oL_{loc}^2(N)$, where $Y \in \{\psi \cdot M : \psi \in L_{loc}^2(M)\}$. Consider the Galtchouk-Kunita-Watanabe decomposition of $H^n \odot N$ with respect to M given by*

$$H^n \odot N = \phi^n \cdot M + L^n, \quad \text{where } \phi^n \in L_{loc}^2(M), \text{ and } L^n \perp M. \quad (3.33)$$

Then, $\phi^n \cdot M$ converges weakly in $\mathcal{M}_{loc}^2(\mathbb{H})$.

Proof. (a) Denote $(\sigma_k)_{k \geq 1}$ the localizing sequence of $H \cdot [M, M]$. Due to Lemma 3.7, we calculate that

$$\begin{aligned} \left\| \left(\sqrt{H}I_{\{H \leq n\}} - \sqrt{H} \right) \odot M^{\sigma_k} \right\|_{\mathcal{M}^2(\mathbb{H})}^2 &= E \left(\left\langle \left(\sqrt{H}I_{\{H \leq n\}} - \sqrt{H} \right) \odot M \right\rangle_{\sigma_k} \right) \\ &= E \left(\left\langle \left(\sqrt{H}I_{\{H > n\}} \right) \odot M \right\rangle_{\sigma_k} \right) \\ &\leq E \left(HI_{\{H > n\}} \cdot [M, M]_{\sigma_k} \right) \rightarrow 0, \text{ as } n \rightarrow +\infty. \end{aligned}$$

- (b) Without loss of generality, we could assume $\sqrt{H} \odot M$ and $\sqrt{H} \odot N$ are both square integrable. We derive from (a) and Kunita-Watanabe inequality that

$$\begin{aligned} &E \left(\left| \left\langle \sqrt{H}I_{\{H \leq n\}} \odot M, \sqrt{H}I_{\{H \leq n\}} \odot N \right\rangle_{\infty} - \left\langle \sqrt{H} \odot M, \sqrt{H} \odot N \right\rangle_{\infty} \right| \right) \\ &\leq E \left(\left| \left\langle \sqrt{H}I_{\{H \leq n\}} \odot M, \sqrt{H}I_{\{H \leq n\}} \odot N \right\rangle_{\infty} - \left\langle \sqrt{H}I_{\{H \leq n\}} \odot M, \sqrt{H} \odot N \right\rangle_{\infty} \right| \right) \\ &+ E \left(\left| \left\langle \sqrt{H}I_{\{H \leq n\}} \odot M, \sqrt{H} \odot N \right\rangle_{\infty} - \left\langle \sqrt{H} \odot M, \sqrt{H} \odot N \right\rangle_{\infty} \right| \right) \\ &= E \left(\left| \left\langle \sqrt{H}I_{\{H \leq n\}} \odot M, \sqrt{H}I_{\{H > n\}} \odot N \right\rangle_{\infty} \right| \right) + E \left(\left| \left\langle \sqrt{H}I_{\{H > n\}} \odot M, \sqrt{H} \odot N \right\rangle_{\infty} \right| \right) \\ &\leq \sqrt{E \left(\left\langle \sqrt{H}I_{\{H \leq n\}} \odot M \right\rangle_{\infty} \right)} \sqrt{E \left(\left\langle \sqrt{H}I_{\{H > n\}} \odot N \right\rangle_{\infty} \right)} \\ &+ \sqrt{E \left(\left\langle \sqrt{H}I_{\{H > n\}} \odot M \right\rangle_{\infty} \right)} \sqrt{E \left(\left\langle \sqrt{H} \odot N \right\rangle_{\infty} \right)} \rightarrow 0, \text{ as } n \rightarrow +\infty. \end{aligned}$$

- (c) For any $\psi \in L_{loc}^2(M)$, using Lemma 3.7, we derive

$$\begin{aligned} \langle \phi^n \cdot M, \psi \cdot M \rangle^{\mathbb{H}} &= \langle H^n \odot N, \psi \cdot M \rangle^{\mathbb{H}} = \left(H^n \cdot [N, \psi \cdot M] \right)^{p, \mathbb{H}} \\ &= \left\langle \sqrt{H^n} \odot N, \psi \sqrt{H^n} \odot M \right\rangle^{\mathbb{H}} + \sum^{p, \mathbb{H}} \left(\sqrt{H^n} \Delta N \right)^{p, \mathbb{H}} \left(\psi \sqrt{H^n} \Delta M \right) \\ &:= V_1^n + V_2^n. \end{aligned} \quad (3.34)$$

Thanks to assertions (a) and (b), we deduce that both processes $\sqrt{H^n} \odot N$ and $\psi\sqrt{H^n} \odot M$ converge weakly in $\mathcal{M}_{loc}^2(\mathbb{H})$ to $\sqrt{H} \odot N$ and $\sqrt{H}\psi \odot M$, and

$$V_1^n \text{ converges locally in } L^1(P) \text{ to } \left\langle \sqrt{H} \odot N, \psi\sqrt{H} \odot M \right\rangle^{\mathbb{H}}. \quad (3.35)$$

Due to Cauchy–Schwarz inequality, we derive

$$\begin{aligned} |V_2^{n+l} - V_2^n| &\leq \sqrt{(HI_{\{n < H \leq n+l\}} \cdot [N])^{p, \mathbb{H}}} \sqrt{(H \cdot [\psi \cdot M])^{p, \mathbb{H}}} \\ &\quad + \sqrt{(HI_{\{n < H \leq n+l\}} \cdot [\psi \cdot M])^{p, \mathbb{H}}} \sqrt{(H \cdot [N])^{p, \mathbb{H}}}. \end{aligned}$$

An application of the Lebesgue Dominating convergence theorem implies the local convergence in $L^1(P)$ of the process V_2^n . This proves that $\langle \phi^n \cdot M, \psi \cdot M \rangle^{\mathbb{H}} = \langle H^n \odot N, \psi \cdot M \rangle^{\mathbb{H}}$ converges locally in $L^1(P)$. Then for any $K \in \mathcal{M}_{loc}^2(\mathbb{H})$, we have

$$K = \theta^K \cdot M + N^K, \quad \text{where } \theta^K \in L_{loc}^2(M) \text{ and } \langle M, N^K \rangle^{\mathbb{H}} = 0.$$

Therefore, $\langle \phi^n \cdot M, K \rangle^{\mathbb{H}} = \langle \phi^n \cdot M, \theta^K \cdot M \rangle^{\mathbb{H}}$ converges locally in $L^1(\mathbb{P})$ and $\phi^n \cdot M$ converges weakly in $\mathcal{M}_{loc}^2(\mathbb{H})$. This completes the proof of the proposition. \square

The following proposition proves that $(\tilde{Z})^{-\frac{1}{2}} I_{[0, \tau]}$ is locally square integrable with respect to a class of \mathbb{G} -local martingales.

Proposition 3.12. *If M is an \mathbb{F} -locally square integrable local martingale, then $I_{[0, \tau]}(\tilde{Z})^{-1} \cdot [\widehat{M}^{(b)}, \widehat{M}^{(b)}] \in \mathcal{A}_{loc}^+(\mathbb{G})$. As a result, $I_{[0, \tau]}(\tilde{Z})^{-\frac{1}{2}}$ belongs to ${}^o L_{loc}^2(\widehat{M}^{(b)}, \mathbb{G})$, where $\widehat{M}^{(b)}$ is given by (3.29).*

Proof. Since $M \in \mathcal{M}_{loc}^2(\mathbb{F})$ and m is bounded, then there exists a sequence of \mathbb{F} -stopping times, $(T_k)_{k \geq 1}$, that increases to $+\infty$ such that

$$\langle M \rangle_{T_k}^{\mathbb{F}} + \text{Var}(\langle M, m \rangle^{\mathbb{F}})_{T_k} \leq k, \quad P - a.s. \quad (3.36)$$

Since $Z_-^{-1} I_{[0, \tau]}$ is locally bounded, then there exists a sequence of \mathbb{G} -stopping times, $(\tau_k)_{k \geq 1}$, that increases to $+\infty$ and

$$\sup_{0 \leq t \leq \tau_k} ((Z_{t-})^{-1} I_{\{t \leq \tau\}}) \leq k, \quad P - a.s. \quad (3.37)$$

Consider

$$\sigma_k := \tau_k \wedge T_k, \quad k \geq 1, \quad \text{and} \quad W := I_{[0, \tau]}(Z_-)^{-1} \cdot \langle M, m \rangle^{\mathbb{F}}. \quad (3.38)$$

Then it is clear that $(\sigma_k)_{k \geq 1}$ is a localizing sequence for the process W , and due to Lemma 3.9-(b), we obtain

$$\begin{aligned} E \left\{ \frac{I_{[0, \tau]}}{\tilde{Z}} \cdot [\widehat{M}^{(b)}]_{\sigma_k} \right\} &\leq 2E \left\{ \frac{I_{[0, \tau]}}{\tilde{Z}} \cdot [M]_{T_k} \right\} + 2E \left\{ \frac{I_{[0, \tau]}}{\tilde{Z}} \cdot [W]_{\sigma_k} \right\} \\ &\leq 2E \left\{ I_{\{\tilde{Z} > 0\}} \cdot [M]_{T_k} \right\} + 2E \left\{ \frac{I_{[0, \tau]}}{Z_-} \cdot [W]_{\sigma_k} \right\} \\ &\leq 2E[M]_{T_k} + 2kE[W]_{\sigma_k} < +\infty. \end{aligned}$$

Hence, $\tilde{Z}^{-1} I_{[0, \tau]} \cdot [\widehat{M}^{(b)}]$ is locally integrable and $I_{[0, \tau]}(\tilde{Z})^{-1/2} \in {}^o L_{loc}^2(\widehat{M}^{(b)}, \mathbb{G})$ (see He et al. [23]). \square

Throughout the rest of this subsection, we will stick to the following notations

$$h^n := I_{\{\tilde{Z} \geq \frac{1}{n}\}}, \quad H^n := Z_- \left(\tilde{Z} \right)^{-1} h^n, \quad n \geq 1. \quad (3.39)$$

As an application of Proposition 3.11 and Proposition 3.12, in the following, we prove that $\phi^n \cdot \widehat{M}^{(b)}$ converges weakly in $\mathcal{M}_{loc}^2(\mathbb{G})$, where $H^n \odot \widehat{m}^{(b)} = \phi^n \cdot \widehat{M}^{(b)} + L^n$, $\phi^n \in L_{loc}^2(\widehat{M}^{(b)})$, and $L^n \perp \widehat{M}^{(b)}$.

Proposition 3.13. *Let $M \in \mathcal{M}_{loc}^2(\mathbb{F})$ and $\widehat{M}^{(b)}$ is given by (3.29). Then the following hold:*

- (a) $\left(I_{[0, \tau]}(\tilde{Z})^{-1/2} I_{\{\tilde{Z} \geq 1/n\}} \odot \widehat{M}^{(b)} \right)$ converges in $\mathcal{M}_{loc}^2(\mathbb{G})$ to $\left(I_{[0, \tau]}(\tilde{Z})^{-1/2} \odot \widehat{M}^{(b)} \right)$.
- (b) For any $K \in \mathcal{M}_{loc}^2(\mathbb{F})$, there exists a sequence of \mathbb{G} -stopping times, $(\eta_k)_{k \geq 1}$ that increases to $+\infty$, and for each $k \geq 1$

$$\left\langle I_{[0, \tau]}(\tilde{Z})^{-1/2} I_{\{\tilde{Z} \geq 1/n\}} \odot \widehat{M}^{(b)}, I_{[0, \tau]}(\tilde{Z})^{-1/2} I_{\{\tilde{Z} \geq 1/n\}} \odot \widehat{K}^{(b)} \right\rangle_{\eta_k}^{\mathbb{G}}$$

converges in $L^1(P)$ to

$$\left\langle I_{[0, \tau]}(\tilde{Z})^{-1/2} \odot \widehat{M}^{(b)}, I_{[0, \tau]}(\tilde{Z})^{-1/2} \odot \widehat{K}^{(b)} \right\rangle_{\eta_k}^{\mathbb{G}},$$

where $\widehat{K}^{(b)}$ is defined via (3.29).

- (c) Consider the Galtchouk-Kunita-Watanabe decomposition of $H^n \odot \widehat{m}^{(b)}$ with respect to $\widehat{M}^{(b)}$

$$H^n \odot \widehat{m}^{(b)} = \phi^n \cdot \widehat{M}^{(b)} + L^n, \quad \text{where } \phi^n \in L_{loc}^2(\widehat{M}^{(b)}), \text{ and } L^n \perp \widehat{M}^{(b)}. \quad (3.40)$$

Then, $\phi^n \cdot \widehat{M}^{(b)}$ converges weakly in $\mathcal{M}_{0, loc}^2(\mathbb{G})$ and

$$\Phi_1 := \lim_{n \rightarrow +\infty} \frac{d\langle H^n \odot \widehat{m}^{(b)}, \widehat{M}^{(b)} \rangle^{\mathbb{G}}}{d\langle \widehat{M}^{(b)}, \widehat{M}^{(b)} \rangle^{\mathbb{G}}} \in L_{loc}^2(\widehat{M}^{(b)}, \mathbb{G}). \quad (3.41)$$

Proof. It is a direct consequence of Proposition 3.11 and Proposition 3.12 by taking $H := I_{[0, \tau]}(\tilde{Z})^{-1}$, $H^n := \tilde{Z}^{-1} I_{[0, \tau]} I_{\{\tilde{Z} \geq \frac{1}{n}\}}$, and $\mathbb{H} = \mathbb{G}$. To complete the proof, we just need to show (3.41). To this end, we apply Proposition 3.11 and Lemma 3.3 to $H^n \odot \widehat{m}^{(b)}$ to conclude that there exists $\Phi_1 \in L_{loc}^2(\widehat{M}^{(b)}, \mathbb{G})$ such that $\langle H^n \odot \widehat{m}^{(b)}, \widehat{M}^{(b)} \rangle^{\mathbb{G}}$ converges locally in L^1 to $\langle \Phi_1 \cdot \widehat{M}^{(b)}, \widehat{M}^{(b)} \rangle^{\mathbb{G}}$ and

$$\Phi_1 := \lim_{n \rightarrow +\infty} \frac{d\langle H^n \odot \widehat{m}^{(b)}, \widehat{M}^{(b)} \rangle^{\mathbb{G}}}{d\langle \widehat{M}^{(b)}, \widehat{M}^{(b)} \rangle^{\mathbb{G}}}. \quad (3.42)$$

This completes the proof of the proposition. \square

As explained in the Remark 3.10, we will characterize the relationship between $I_{[0, \tau]} \cdot \langle m, M \rangle^{\mathbb{F}}$ and $\langle \widehat{M}^{(b)}, \widehat{M}^{(b)} \rangle^{\mathbb{G}}$. This is the main focus in the following.

Proposition 3.14. *Let $M \in \mathcal{M}_{loc}^2(\mathbb{F})$ and $\widehat{M}^{(b)}$ is given in (3.29). If $\{\Delta M \neq 0\} \cap \{0 = \tilde{Z} < Z_-\} = \emptyset$, we have*

$$\frac{1}{Z_-} I_{[0, \tau]} \cdot \langle m, M \rangle^{\mathbb{F}} = \widehat{\Phi}_1 \cdot \langle \widehat{M}^{(b)} \rangle^{\mathbb{G}}, \quad \text{and } \widehat{\Phi}_1 \in L_{loc}^2(\widehat{M}^{(b)}, \mathbb{G}), \quad (3.43)$$

where $\widehat{\Phi}_1 := \Phi_1 \left({}^p\mathbb{F} \left(I_{\{\tilde{Z} > 0\}} \right) (Z_-^2 + \Delta \langle m \rangle^{\mathbb{F}}) \right)^{-1} Z_-^2 I_{[0, \tau]}$ and Φ_1 is given in (3.41).

Proof. By Proposition 3.13, we know that

$$\lim_n \langle H^n \odot \widehat{m}^{(b)}, \widehat{M}^{(b)} \rangle^{\mathbb{G}} = \Phi_1 \cdot \langle \widehat{M}^{(b)}, \widehat{M}^{(b)} \rangle^{\mathbb{G}}, \quad (3.44)$$

where H^n and $\widehat{m}^{(b)}$ are defined in (3.39) and (3.29) respectively. Now it remains to describe explicitly the limit $\lim_n \langle H^n \odot \widehat{m}^{(b)}, \widehat{M}^{(b)} \rangle^{\mathbb{G}}$. To this end, we first calculate

$$\begin{aligned} \frac{1}{Z_-} \cdot [\widehat{m}^{(b)}, \widehat{M}^{(b)}] &= \frac{1}{Z_-} I_{[0,\tau]} \cdot [m, M] - I_{[0,\tau]} \frac{\Delta M}{Z_-^2} \cdot \langle m \rangle^{\mathbb{F}} - \frac{\Delta m}{Z_-^2} I_{[0,\tau]} \cdot \langle M, m \rangle^{\mathbb{F}} \\ &\quad + \frac{1}{Z_-^3} I_{[0,\tau]} \Delta \langle m \rangle^{\mathbb{F}} \cdot \langle M, m \rangle^{\mathbb{F}}. \end{aligned} \quad (3.45)$$

Then, by integrating H^n on both sides above, and using the properties of optional integration (see Proposition 3.5 and Proposition 3.11), we obtain

$$\begin{aligned} \frac{1}{Z_-} I_{[0,\tau]} \cdot (h^n \cdot [m, M])^{p,\mathbb{F}} &= \frac{1}{Z_-} I_{[0,\tau]} \cdot (H^n \cdot [m, M])^{p,\mathbb{G}} \\ &= \left(\frac{1}{Z_-} H^n \cdot [\widehat{m}^{(b)}, \widehat{M}^{(b)}] \right)^{p,\mathbb{G}} + I_{[0,\tau]} \frac{p,\mathbb{G} \left(H^n \Delta M \right)}{Z_-^2} \cdot \langle m \rangle^{\mathbb{F}} \\ &\quad + I_{[0,\tau]} \frac{p,\mathbb{G} \left(H^n \Delta m \right)}{Z_-^2} \cdot \langle M, m \rangle^{\mathbb{F}} - I_{[0,\tau]} \frac{\Delta \langle M, m \rangle^{\mathbb{F}}}{Z_-^3} p,\mathbb{G} \left(H^n \right) \cdot \langle m \rangle^{\mathbb{F}}. \end{aligned} \quad (3.46)$$

Due to $\{\Delta M \neq 0\} \cap \{0 = \widetilde{Z} < Z_-\} = \emptyset$, we get

$$p,\mathbb{F} \left(\Delta M I_{\{\widetilde{Z} > 0\}} \right) = p,\mathbb{F} (\Delta M) = 0.$$

By taking the limit, we derive

$$\begin{aligned} \lim_n \frac{I_{[0,\tau]}}{Z_-} \cdot (h^n \cdot [m, M])^{p,\mathbb{F}} &= \frac{I_{[0,\tau]}}{Z_-} \cdot \left(I_{\{\widetilde{Z} > 0\}} \cdot [m, M] \right)^{p,\mathbb{F}} = \frac{I_{[0,\tau]}}{Z_-} \cdot \langle m, M \rangle^{\mathbb{F}}, \\ \lim_n I_{[0,\tau]} \frac{p,\mathbb{G} \left(H^n \Delta M \right)}{Z_-^2} \cdot \langle m \rangle^{\mathbb{F}} &= I_{[0,\tau]} \frac{-p,\mathbb{F} \left(\Delta M I_{\{\widetilde{Z} = 0\}} \right)}{Z_-^2} \cdot \langle m \rangle^{\mathbb{F}} = 0, \end{aligned} \quad (3.47)$$

and

$$\begin{aligned} \lim_n p,\mathbb{G} \left(H^n \right) I_{[0,\tau]} &= p,\mathbb{F} \left(I_{\{\widetilde{Z} > 0\}} \right) I_{[0,\tau]}, \\ \lim_n I_{[0,\tau]} \frac{p,\mathbb{G} \left(H^n \Delta m \right)}{Z_-^2} &= I_{[0,\tau]} \frac{p,\mathbb{F} \left(\Delta m I_{\{\widetilde{Z} > 0\}} \right)}{Z_-^2} = I_{[0,\tau]} \frac{p,\mathbb{F} \left(I_{\{\widetilde{Z} = 0\}} \right)}{Z_-}, \end{aligned} \quad (3.48)$$

where in (3.46)-(3.48) we used Lemma 3.9. Then, by combining the above equalities and (3.41), we conclude that

$$\Phi_1 \cdot \langle \widehat{M}^{(b)} \rangle^{\mathbb{G}} = \lim_n \left(\frac{1}{Z_-} H^n \cdot [\widehat{m}^{(b)}, \widehat{M}^{(b)}] \right)^{p,\mathbb{G}} = \frac{I_{[0,\tau]}}{Z_-} p,\mathbb{F} \left(I_{\{\widetilde{Z} > 0\}} \right) \left(1 + \frac{\Delta \langle m \rangle^{\mathbb{F}}}{Z_-^2} \right) \cdot \langle M, m \rangle^{\mathbb{F}}.$$

The proof of the proposition is completed due to the \mathbb{G} -local boundedness of $\left(p,\mathbb{F} \left(I_{\{\widetilde{Z} > 0\}} \right) \right)^{-1} I_{[0,\tau]}$ (see Lemma 3.15 below). \square

Lemma 3.15. *The following process*

$$V^{(b)} := \left({}^{p, \mathbb{F}} \left(I_{\{\tilde{Z} > 0\}} \right) \right)^{-1} I_{\llbracket 0, \tau \rrbracket} \quad (3.49)$$

is \mathbb{G} -predictable and locally bounded.

Proof. It is enough to notice that $\tilde{Z} \leq I_{\{\tilde{Z} > 0\}}$ and the process $(Z_-)^{-1} I_{\llbracket 0, \tau \rrbracket}$ is \mathbb{G} -locally bounded. \square

3.3 The key stochastic results for the part after an honest time

In this section, we will present some crucial lemmas and propositions that we need to prove Theorem 2.10.

Lemma 3.16. *For any \mathbb{F} -local martingale M , we associate $\widehat{M}^{(a)}$ given by,*

$$\widehat{M}^{(a)} := I_{\llbracket \tau, +\infty \rrbracket} \cdot M + I_{\llbracket \tau, +\infty \rrbracket} (1 - Z_-)^{-1} \cdot \langle M, m \rangle^{\mathbb{F}}, \quad (3.50)$$

which is a \mathbb{G} -local martingale.

Proof. The proof can be found in [7], [17] and [29]. \square

Similar as on $\llbracket 0, \tau \rrbracket$, we will keep the notation $\widehat{M}^{(a)}$ defined in (3.50) for any \mathbb{F} -local martingale.

Below, we recall an important lemma due to Choulli et al. [12].

Lemma 3.17. *Suppose that $Z_\tau < 1$. Then the following assertions hold.*

- (a) *The process $(1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket}$ is a \mathbb{G} -locally bounded and predictable process.*
- (b) *For any \mathbb{F} -adapted process with locally integrable variation, V , we have*

$$I_{\llbracket \tau, +\infty \rrbracket} \cdot V^{p, \mathbb{G}} = I_{\llbracket \tau, +\infty \rrbracket} (1 - Z_-)^{-1} \cdot \left((1 - \tilde{Z}) \cdot V \right)^{p, \mathbb{F}}. \quad (3.51)$$

- (c) *For any process V as in (b), the \mathbb{G} -predictable projection of ΔV , is given on $\llbracket \tau, +\infty \rrbracket$ by*

$${}^{p, \mathbb{G}} (\Delta V) = (1 - Z_-)^{-1} {}^{p, \mathbb{F}} \left((1 - \tilde{Z}) \Delta V \right). \quad (3.52)$$

- (d) *For any \mathbb{F} -local martingale, on $\llbracket \tau, +\infty \rrbracket$, we have*

$${}^{p, \mathbb{G}} \left(\frac{\Delta M}{1 - \tilde{Z}} \right) = \frac{{}^{p, \mathbb{F}} \left(\Delta M I_{\{\tilde{Z} < 1\}} \right)}{1 - Z_-}, \quad \text{and} \quad {}^{p, \mathbb{G}} \left(\frac{1}{1 - \tilde{Z}} \right) = \frac{{}^{p, \mathbb{F}} \left(I_{\{\tilde{Z} < 1\}} \right)}{1 - Z_-}. \quad (3.53)$$

The following proposition proves that $(1 - \tilde{Z})^{-\frac{1}{2}} I_{\llbracket \tau, +\infty \rrbracket}$ is locally square integrable with respect to a class of \mathbb{G} -local martingales.

Proposition 3.18. *Let M be an \mathbb{F} -locally square integrable local martingale, then*

$$I_{\llbracket \tau, +\infty \rrbracket} \left((1 - \tilde{Z})^{-1} \cdot [\widehat{M}^{(a)}] \right) \in \mathcal{A}_{loc}^+(\mathbb{G}),$$

where $\widehat{M}^{(a)}$ is defined via (3.50). As a result, $I_{\llbracket \tau, +\infty \rrbracket} \left((1 - \tilde{Z})^{-1/2} \right) \in {}^o L_{loc}^2(\widehat{M}^{(a)}, \mathbb{G})$.

Proof. Let us denote the localizing sequences of $W := I_{\llbracket \tau, +\infty \rrbracket} (1 - Z_-)^{-1/2} \cdot \langle M, m \rangle^{\mathbb{F}}, (1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket}$ and $[M]$ by $(\sigma_n)_{n \geq 1}$, $(\tau_n)_{n \geq 1}$ and $(T_n)_{n \geq 1}$ respectively. Then, due to Lemma 3.17, we derive that

$$\begin{aligned} E \left\{ \frac{I_{\llbracket \tau, +\infty \rrbracket}}{1 - \tilde{Z}} \cdot [\widehat{M}^{(a)}]_{\sigma_n \wedge \tau_n \wedge T_n} \right\} &\leq 2E \left\{ \frac{I_{\llbracket \tau, +\infty \rrbracket}}{1 - \tilde{Z}} \cdot [M]_{T_n} \right\} + 2E \left\{ \frac{I_{\llbracket \tau, +\infty \rrbracket}}{1 - \tilde{Z}} \cdot [W]_{\sigma_n \wedge \tau_n} \right\} \\ &\leq 2E[M]_{T_n} + E \left\{ \frac{2I_{\llbracket \tau, +\infty \rrbracket}}{1 - \tilde{Z}} \cdot [W]_{\sigma_n \wedge \tau_n} \right\} < +\infty. \end{aligned}$$

This ends the proof of the proposition. \square

Throughout the rest of this subsection, we will use the following notations

$$k^n := I_{\{1 - \tilde{Z} \geq \frac{1}{n}\}}, \quad K^n := (1 - Z_-) \left(1 - \tilde{Z}\right)^{-1} k^n, \quad n \geq 1. \quad (3.54)$$

As a counterpart of Proposition 3.13, we have on $\llbracket \tau, +\infty \rrbracket$:

Proposition 3.19. *Let $M \in \mathcal{M}_{loc}^2(\mathbb{F})$, $\widehat{M}^{(a)}$ defined by (3.50) and*

$$U^n := K^n (1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket}, \quad n \geq 1.$$

Then, the following assertions hold.

- (a) $\left(\sqrt{U^n} \odot \widehat{M}^{(a)}\right)$ converges to $\left(\frac{I_{\llbracket \tau, +\infty \rrbracket}}{\sqrt{1 - \tilde{Z}}} \odot \widehat{M}^{(a)}\right)$ in $\mathcal{M}_{loc}^2(\mathbb{G})$.
- (b) For any $L \in \mathcal{M}_{loc}^2(\mathbb{F})$, there exists a sequence of \mathbb{G} -stopping times $(\eta_k)_{k \geq 1}$ increasing to infinity such that for all k , $\left\langle \sqrt{U^n} \odot \widehat{M}^{(a)}, \sqrt{U^n} \odot \widehat{L} \right\rangle_{\eta_k}^{\mathbb{G}}$ converges in $L^1(P)$ to $\left\langle \frac{I_{\llbracket \tau, +\infty \rrbracket}}{\sqrt{1 - \tilde{Z}}} \odot \widehat{M}^{(a)}, \frac{I_{\llbracket \tau, +\infty \rrbracket}}{\sqrt{1 - \tilde{Z}}} \odot \widehat{L} \right\rangle_{\eta_k}^{\mathbb{G}}$.

Proof. It is the same as the proof of Proposition 3.11. Indeed, it is enough to consider the \mathbb{G} -locally bounded process $(1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket}$, and use the same techniques as in Proposition 3.11. \square

Lemma 3.20. *Under the condition $Z_\tau < 1$, the following process*

$$V^{(a)} := \left({}^{p, \mathbb{F}} \left(I_{\{\tilde{Z} < 1\}} \right) \right)^{-1} I_{\llbracket \tau, +\infty \rrbracket} \quad (3.55)$$

is \mathbb{G} -predictable and locally bounded.

Proof. It is enough to notice that $1 - \tilde{Z} \leq I_{\{\tilde{Z} < 1\}}$ and the process $(1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket}$ is \mathbb{G} -locally bounded. \square

Proposition 3.21. *Let $M \in \mathcal{M}_{loc}^2(\mathbb{F})$ and $\widehat{M}^{(a)}$ is given by (3.50). Then, the following hold.*

- (a) *Consider the Galtchouk-Kunita-Watanabe decomposition of $K^n \odot \widehat{m}^{(a)}$ with respect to $\widehat{M}^{(a)}$*

$$K^n \odot \widehat{m}^{(a)} = \theta^n \cdot \widehat{M}^{(a)} + L^n, \quad \text{where } \theta^n \in L_{loc}^2(\widehat{M}^{(a)}), \text{ and } L^n \perp \widehat{M}^{(a)}. \quad (3.56)$$

Then, $(\theta^n \cdot \widehat{M}^{(a)})$ converges weakly in $\mathcal{M}_{loc}^2(\mathbb{G})$.

As a result, we have

$$\Phi_2 := \lim_{n \rightarrow +\infty} \frac{d\langle K^n \odot \widehat{m}^{(a)}, \widehat{M}^{(a)} \rangle_{\mathbb{G}}}{d\langle \widehat{M}^{(a)}, \widehat{M}^{(a)} \rangle_{\mathbb{G}}} \in L_{loc}^2(\widehat{M}^{(a)}, \mathbb{G}). \quad (3.57)$$

(b) If $\{\Delta M \neq 0\} \cap \{1 = \tilde{Z} > Z_-\} = \emptyset$, we have

$$\frac{1}{1 - Z_-} I_{\tau, +\infty} \llbracket \cdot \langle m, M \rangle^{\mathbb{F}} = \widehat{\Phi}_2 I_{\llbracket 0, \tau \rrbracket} \llbracket \cdot \widehat{M}^{(a)} \rangle^{\mathbb{G}}, \quad \text{and} \quad \widehat{\Phi}_2 \in L_{loc}^2(\widehat{M}^{(a)}), \quad (3.58)$$

where

$$\widehat{\Phi}_2 := \Phi_2 \left({}^{p, \mathbb{F}} \left(I_{\{\tilde{Z} < 1\}} \right) \left(1 + \frac{\Delta \langle m \rangle^{\mathbb{F}}}{(1 - Z_-)^2} \right) \right)^{-1} I_{\tau, +\infty} \llbracket, \quad \text{and} \quad \Phi_2 \text{ is given in (3.57).}$$

Proof. (a) It is a consequence of Proposition 3.11 and Lemma 3.3 by taking

$$K^n := (1 - Z_-) \left(1 - \tilde{Z} \right)^{-1} I_{\{1 - \tilde{Z} \geq \frac{1}{n}\}}, \quad \text{and} \quad \mathbb{H} = \mathbb{G}. \quad (3.59)$$

(b) Now it remains to describe explicitly the limit in (3.57), that is $\lim_n \langle K^n \odot \widehat{m}^{(a)}, \widehat{M}^{(a)} \rangle^{\mathbb{G}}$. To this end, we first calculate

$$\begin{aligned} \frac{1}{1 - Z_-} \llbracket \widehat{m}^{(a)}, \widehat{M}^{(a)} \rrbracket &= \frac{1}{Z_-} I_{\tau, +\infty} \llbracket \cdot [m, M] + I_{\tau, +\infty} \llbracket \frac{\Delta M}{(1 - Z_-)^2} \cdot \langle m \rangle^{\mathbb{F}} + \frac{\Delta m}{(1 - Z_-)^2} I_{\tau, +\infty} \llbracket \cdot \langle M, m \rangle^{\mathbb{F}} \\ &\quad + \frac{1}{(1 - Z_-)^3} I_{\tau, +\infty} \llbracket \Delta \langle m \rangle^{\mathbb{F}} \cdot \langle M, m \rangle^{\mathbb{F}}. \end{aligned} \quad (3.60)$$

Then by integrating K^n on both sides above, and using the properties of optional integration (see Proposition 3.5 and Proposition 3.17), we obtain

$$\begin{aligned} \frac{1}{1 - Z_-} I_{\tau, +\infty} \llbracket \cdot (k^n \cdot [m, M])^{p, \mathbb{F}} &= \frac{1}{1 - Z_-} I_{\tau, +\infty} \llbracket \cdot (K^n \cdot [m, M])^{p, \mathbb{G}} \\ &= \left(\frac{1}{1 - Z_-} K^n \cdot [\widehat{m}^{(a)}, \widehat{M}^{(a)}] \right)^{p, \mathbb{G}} - I_{\tau, +\infty} \llbracket \frac{{}^{p, \mathbb{G}}(K^n \Delta M)}{(1 - Z_-)^2} \cdot \langle m \rangle^{\mathbb{F}} \\ &\quad - I_{\tau, +\infty} \llbracket \frac{{}^{p, \mathbb{G}}(K^n \Delta m)}{(1 - Z_-)^2} \cdot \langle M, m \rangle^{\mathbb{F}} - I_{\tau, +\infty} \llbracket \frac{\Delta \langle M, m \rangle^{\mathbb{F}}}{(1 - Z_-)^3} {}^{p, \mathbb{G}}(K^n) \cdot \langle m \rangle^{\mathbb{F}}. \end{aligned} \quad (3.61)$$

Due to $\{\Delta M \neq 0\} \cap \{1 = \tilde{Z} > Z_-\} = \emptyset$, we have

$${}^{p, \mathbb{F}} \left(\Delta M I_{\{\tilde{Z} < 1\}} \right) I_{\{Z_- < 1\}} = {}^{p, \mathbb{F}} (\Delta M) I_{\{Z_- < 1\}} = 0.$$

By taking the limit, we derive

$$\begin{aligned} \lim_n \frac{I_{\tau, +\infty} \llbracket \cdot (k^n \cdot [m, M])^{p, \mathbb{F}}}{1 - Z_-} &= \frac{I_{\tau, +\infty} \llbracket \cdot \left(I_{\{\tilde{Z} < 1\}} \cdot [m, M] \right)^{p, \mathbb{F}}}{1 - Z_-} = \frac{I_{\tau, +\infty} \llbracket \cdot \langle m, M \rangle^{\mathbb{F}}}{1 - Z_-}, \\ \lim_n I_{\tau, +\infty} \llbracket \frac{{}^{p, \mathbb{G}}(K^n \Delta M)}{(1 - Z_-)^2} \cdot \langle m \rangle^{\mathbb{F}} &= I_{\tau, +\infty} \llbracket \frac{{}^{-p, \mathbb{F}}(\Delta M I_{\{\tilde{Z} = 1\}})}{(1 - Z_-)^2} \cdot \langle m \rangle^{\mathbb{F}} = 0, \end{aligned} \quad (3.62)$$

and

$$\begin{aligned} \lim_n {}^{p, \mathbb{G}}(K^n) I_{\tau, +\infty} \llbracket &= {}^{p, \mathbb{F}} \left(I_{\{\tilde{Z} < 1\}} \right) I_{\tau, +\infty} \llbracket, \\ \lim_n I_{\tau, +\infty} \llbracket \frac{{}^{p, \mathbb{G}}(K^n \Delta m)}{(1 - Z_-)^2} &= I_{\tau, +\infty} \llbracket \frac{{}^{p, \mathbb{F}}(\Delta m I_{\{\tilde{Z} < 1\}})}{(1 - Z_-)^2} = -I_{\tau, +\infty} \llbracket \frac{{}^{p, \mathbb{F}}(I_{\{\tilde{Z} = 1\}})}{1 - Z_-}. \end{aligned} \quad (3.63)$$

Then, by combining (3.57), (3.61) and (3.62), we conclude that

$$\begin{aligned}\Phi_1 \cdot \langle \widehat{M}^{(a)} \rangle^{\mathbb{G}} &= \lim_n \left(\frac{1}{1 - Z_-} K^n \cdot [\widehat{m}^{(b)}, \widehat{M}^{(a)}] \right)^{p, \mathbb{G}} \\ &= \frac{I_{\tau, +\infty}}{1 - Z_-} \cdot^{p, \mathbb{F}} \left(I_{\{\tilde{Z} < 1\}} \right) \left(1 + \frac{\Delta \langle m \rangle^{\mathbb{F}}}{(1 - Z_-)^2} \right) \cdot \langle M, m \rangle^{\mathbb{F}}.\end{aligned}$$

The proof of the proposition is completed. \square

4 Proof Theorems 2.4 and 2.10

Now, we have prepared all the ingredients to prove the two main theorems in the Section 2 (Theorem 2.4 and Theorem 2.10).

4.1 Proof Theorem 2.4

Suppose that S satisfies Structure Conditions under \mathbb{F} . Then, there exist a locally square integrable \mathbb{F} -local martingale, M^S , and an \mathbb{F} -predictable process $\widehat{\lambda} \in L_{loc}^2(M^S, \mathbb{F})$ such that

$$S = S_0 + M^S + A^S = S_0 + M^S - \widehat{\lambda} \cdot \langle M^S \rangle^{\mathbb{F}}. \quad (4.64)$$

For notational simplicity, we put $M = M^S$ and $\widehat{M}^{(b)} = \widehat{M}^{S(b)}$, where $\widehat{M}^{S(b)}$ is defined via (3.29). Then the canonical decomposition of S^τ under \mathbb{G} has the form of

$$S^\tau = S_0 + M^\tau - \widehat{\lambda} I_{[0, \tau]} \cdot \langle M \rangle^{\mathbb{F}} =: S_0 + \widehat{M}^{(b)} + \frac{1}{Z_-} I_{[0, \tau]} \cdot \langle M, m \rangle^{\mathbb{F}} - \widehat{\lambda} I_{[0, \tau]} \cdot \langle M \rangle^{\mathbb{F}}. \quad (4.65)$$

Recall the notations in (3.39),

$$h^n := I_{\{\tilde{Z} \geq \frac{1}{n}\}}, \quad H^n := Z_- \left(\tilde{Z} \right)^{-1} h^n, \quad n \geq 1.$$

Consider the following locally square integrable \mathbb{G} -local martingale.

$$\widetilde{N}^{(b)} := \frac{1}{Z_-} \cdot \widehat{m}^{(b)} - \widehat{\lambda} \cdot \widehat{M}^{(b)}. \quad (4.66)$$

Notice that

$$\begin{aligned}& \frac{1}{Z_-} I_{[0, \tau]} \cdot [m, M] - \widehat{\lambda} I_{[0, \tau]} \cdot [M] = [\widetilde{N}^{(b)}, \widehat{M}^{(b)}] + \frac{\Delta m}{Z_-^2} I_{[0, \tau]} \cdot \langle M, m \rangle^{\mathbb{F}} \\ & + \frac{\Delta M}{Z_-^2} I_{[0, \tau]} \cdot \langle m \rangle^{\mathbb{F}} - \frac{\Delta \langle M, m \rangle^{\mathbb{F}}}{Z_-^3} I_{[0, \tau]} \cdot \langle m \rangle^{\mathbb{F}} - \frac{2\widehat{\lambda}}{Z_-} \Delta M \cdot \langle M, m \rangle^{\mathbb{F}} + \frac{\widehat{\lambda}}{Z_-^2} \Delta \langle M, m \rangle^{\mathbb{F}} \cdot \langle M, m \rangle^{\mathbb{F}}.\end{aligned}$$

Then, by integrating both sides with H^n and combining the obtained equality with the properties of the optional integral (see Proposition 3.5 and Lemma 3.9), we derive that

$$\begin{aligned}& \frac{1}{Z_-} I_{[0, \tau]} \cdot (h^n \cdot [m, M])^{p, \mathbb{F}} - \widehat{\lambda} I_{[0, \tau]} \cdot (h^n \cdot [M])^{p, \mathbb{F}} \\ & = \frac{1}{Z_-} I_{[0, \tau]} \cdot (H^n \cdot [m, M])^{p, \mathbb{G}} - \widehat{\lambda} I_{[0, \tau]} \cdot (H^n \cdot [M])^{p, \mathbb{G}}\end{aligned}$$

$$\begin{aligned}
&= \left(H^n \cdot [\tilde{N}^{(b)}, \widehat{M}^{(b)}] \right)^{p, \mathbb{G}} + \frac{p, \mathbb{G} \left(H^n \Delta M \right)}{(Z_-)^2} I_{[0, \tau]} \cdot \langle m \rangle^{\mathbb{F}} - \frac{\Delta \langle M, m \rangle^{\mathbb{F}} \, p, \mathbb{G} \left(H^n \right)}{(Z_-)^3} I_{[0, \tau]} \cdot \langle m \rangle^{\mathbb{F}} \\
&\quad - \frac{2\widehat{\lambda}}{Z_-} \, p, \mathbb{G} \left(H^n \Delta M \right) I_{[0, \tau]} \cdot \langle M, m \rangle^{\mathbb{F}} + \frac{\widehat{\lambda} \Delta \langle M, m \rangle^{\mathbb{F}}}{(Z_-)^2} \, p, \mathbb{G} \left(H^n \right) I_{[0, \tau]} \cdot \langle M, m \rangle^{\mathbb{F}} \\
&\quad + \frac{p, \mathbb{G} \left(H^n \Delta m \right)}{(Z_-)^2} I_{[0, \tau]} \cdot \langle M, m \rangle^{\mathbb{F}}.
\end{aligned} \tag{4.67}$$

Then, the similar arguments as the limits in (3.47) lead to conclude that

$$\lim_n \left(H^n \cdot [\tilde{N}^{(b)}, \widehat{M}^{(b)}] \right)^{p, \mathbb{G}} = I_{[0, \tau]} \frac{1}{Z_-} \cdot \langle M, m \rangle^{\mathbb{F}} - \widehat{\lambda} I_{[0, \tau]} \cdot \langle M \rangle^{\mathbb{F}} + R^{(b)} \cdot \langle M, m \rangle^{\mathbb{F}},$$

where

$$R^{(b)} := \frac{1}{Z_-} \left(- \, p, \mathbb{F} (I_{\{\tilde{Z}=0\}}) + \frac{\Delta \langle m \rangle^{\mathbb{F}}}{Z_-^2} \, p, \mathbb{F} (I_{\{\tilde{Z}>0\}}) - \frac{\widehat{\lambda} \Delta \langle M, m \rangle^{\mathbb{F}}}{Z_-} \, p, \mathbb{F} (I_{\{\tilde{Z}>0\}}) \right) I_{[0, \tau]}.$$

Again, by applying Proposition 3.11 to $H^n \odot \tilde{N}^{(b)}$ and $\widehat{M}^{(b)}$, we conclude that there exists $\Phi^{(b)} \in L_{loc}^2(\widehat{M}^{(b)}, \mathbb{G})$ such that

$$\langle H^n \odot \tilde{N}^{(b)}, \widehat{M}^{(b)} \rangle^{\mathbb{G}} \text{ converges locally in } L^1 \text{ to } \langle \Phi^{(b)}, \widehat{M}^{(b)} \rangle^{\mathbb{G}}.$$

Recall that $I_{[0, \tau]} \cdot \langle M, m \rangle^{\mathbb{F}} = \widehat{\Phi}_1 \cdot \langle \widehat{M}^{(b)} \rangle^{\mathbb{G}}$ in (3.58). Thus, the uniqueness of the limit leads to

$$I_{[0, \tau]} \frac{1}{Z_-} \cdot \langle M, m \rangle^{\mathbb{F}} - \widehat{\lambda} I_{[0, \tau]} \cdot \langle M \rangle^{\mathbb{F}} = \left(\Phi^{(b)} - R^{(b)} \widehat{\Phi}_1 \right) \cdot \langle \widehat{M}^{(b)} \rangle^{\mathbb{G}}. \tag{4.68}$$

Due to the locally boundedness of $Z_-^{-1} I_{[0, \tau]}$, it is easy to see that

$$\widehat{\lambda}^{\mathbb{G}} := \left(\Phi^{(b)} - R^{(b)} \widehat{\Phi}_1 \right) \in L_{loc}^2(\widehat{M}^{(b)}, \mathbb{G}), \tag{4.69}$$

and

$$S^\tau = S_0 + \widehat{M}^{(b)} + \widehat{\lambda}^{\mathbb{G}} \cdot \langle \widehat{M}^{(b)}, \widehat{M}^{(b)} \rangle^{\mathbb{G}}. \tag{4.70}$$

This proves the Structure Conditions for S^τ under \mathbb{G} , and the proof of the theorem is completed. \square

4.2 Proof Theorem 2.10

Suppose that S satisfies Structure Condition under \mathbb{F} . Then, there exist a locally square integrable \mathbb{F} -local martingale, M^S , and an \mathbb{F} predictable process $\widehat{\lambda} \in L_{loc}^2(M^S, \mathbb{F})$ such that

$$S = S_0 + M^S + A^S = S_0 + M^S - \widehat{\lambda} \cdot \langle M^S \rangle^{\mathbb{F}}. \tag{4.71}$$

For notational simplicity, we put $M = M^S$ and $\widehat{M}^{(a)} = \widehat{M}^S^{(a)}$, where $\widehat{M}^S^{(a)}$ is defined via (3.50). Then we get

$$\begin{aligned}
I_{[\tau, +\infty]} \cdot S &= I_{[\tau, +\infty]} \cdot M - \widehat{\lambda} I_{[\tau, +\infty]} \cdot \langle M \rangle^{\mathbb{F}} \\
&=: \widehat{M}^{(a)} - \frac{1}{1 - Z_-} I_{[\tau, +\infty]} \cdot \langle M, m \rangle^{\mathbb{F}} - \widehat{\lambda} I_{[\tau, +\infty]} \cdot \langle M \rangle^{\mathbb{F}}.
\end{aligned} \tag{4.72}$$

Recall the notations in (3.54),

$$k^n := I_{\{1-\tilde{Z} \geq \frac{1}{n}\}}, \quad K^n := (1 - Z_-) \left(1 - \tilde{Z}\right)^{-1} k^n, \quad n \geq 1.$$

Consider the following locally square integrable \mathbb{G} -local martingale.

$$\tilde{N}^{(a)} := -\frac{1}{1 - Z_-} \cdot \widehat{m}^{(a)} - \widehat{\lambda} \cdot \widehat{M}^{(a)}, \quad (4.73)$$

where $\widehat{m}^{(a)}$ is given by (3.50). Notice that

$$\begin{aligned} -\frac{1}{1 - Z_-} I_{\tau, +\infty} \llbracket \cdot [m, M] \rrbracket - \widehat{\lambda} I_{\tau, +\infty} \llbracket \cdot [M] \rrbracket &= [\tilde{N}^{(a)}, \widehat{M}^{(a)}] + \frac{\Delta m}{(1 - Z_-)^2} I_{\tau, +\infty} \llbracket \cdot \langle M, m \rangle^{\mathbb{F}} \rrbracket \\ &+ \frac{\Delta M}{(1 - Z_-)^2} I_{\tau, +\infty} \llbracket \cdot \langle m \rangle^{\mathbb{F}} \rrbracket + \frac{\Delta \langle M, m \rangle^{\mathbb{F}}}{(1 - Z_-)^3} I_{\tau, +\infty} \llbracket \cdot \langle m \rangle^{\mathbb{F}} \rrbracket \\ &+ \frac{2\widehat{\lambda}}{1 - Z_-} \Delta M \cdot \langle M, m \rangle^{\mathbb{F}} + \frac{\widehat{\lambda}}{(1 - Z_-)^2} \Delta \langle M, m \rangle^{\mathbb{F}} \cdot \langle M, m \rangle^{\mathbb{F}}. \end{aligned}$$

Then, by integrating both sides with K^n and combining the obtained equality with the properties of the optional integral (see Proposition 3.5 and Lemma 3.17), we derive

$$\begin{aligned} &-\frac{1}{1 - Z_-} I_{\tau, +\infty} \llbracket \cdot (k^n \cdot [m, M])^{p, \mathbb{F}} - \widehat{\lambda} I_{\tau, +\infty} \llbracket \cdot (k^n \cdot [M])^{p, \mathbb{F}} \rrbracket \\ &= -\frac{1}{1 - Z_-} I_{\tau, +\infty} \llbracket \cdot (K^n \cdot [m, M])^{p, \mathbb{G}} - \widehat{\lambda} I_{\tau, +\infty} \llbracket \cdot (K^n \cdot [M])^{p, \mathbb{G}} \rrbracket \\ &= \left(K^n \cdot [\tilde{N}^{(a)}, \widehat{M}^{(a)}] \right)^{p, \mathbb{G}} + \frac{p, \mathbb{G} \left(K^n \Delta M \right)}{(1 - Z_-)^2} I_{\tau, +\infty} \llbracket \cdot \langle m \rangle^{\mathbb{F}} \rrbracket + \frac{\Delta \langle M, m \rangle^{\mathbb{F}} p, \mathbb{G} \left(K^n \right)}{(1 - Z_-)^3} I_{\tau, +\infty} \llbracket \cdot \langle m \rangle^{\mathbb{F}} \rrbracket \\ &+ \frac{2\widehat{\lambda}}{1 - Z_-} p, \mathbb{G} \left(K^n \Delta M \right) I_{\tau, +\infty} \llbracket \cdot \langle M, m \rangle^{\mathbb{F}} \rrbracket + \frac{\widehat{\lambda} \Delta \langle M, m \rangle^{\mathbb{F}} p, \mathbb{G} \left(K^n \right)}{(1 - Z_-)^2} I_{\tau, +\infty} \llbracket \cdot \langle M, m \rangle^{\mathbb{F}} \rrbracket \\ &+ \frac{p, \mathbb{G} \left(K^n \Delta m \right)}{(1 - Z_-)^2} I_{\tau, +\infty} \llbracket \cdot \langle M, m \rangle^{\mathbb{F}} \rrbracket. \end{aligned} \quad (4.74)$$

Then, the similar arguments as the limits in (3.62) lead to conclude that

$$\lim_n \left(K^n \cdot [\tilde{N}^{(a)}, \widehat{M}^{(a)}] \right)^{p, \mathbb{G}} = I_{\tau, +\infty} \llbracket \frac{-1}{1 - Z_-} \cdot \langle M, m \rangle^{\mathbb{F}} - \widehat{\lambda} I_{\tau, +\infty} \llbracket \cdot \langle M \rangle^{\mathbb{F}} \rrbracket + R^{(a)} \cdot \langle M, m \rangle^{\mathbb{F}},$$

where

$$R^{(a)} := \frac{-1}{1 - Z_-} \left(-p, \mathbb{F} (I_{\{\tilde{Z}=1\}}) + \frac{\Delta \langle m \rangle^{\mathbb{F}}}{(1 - Z_-)^2} p, \mathbb{F} (I_{\{\tilde{Z}<1\}}) + \frac{\widehat{\lambda} \Delta \langle M, m \rangle^{\mathbb{F}}}{1 - Z_-} p, \mathbb{F} (I_{\{\tilde{Z}<1\}}) \right) I_{\tau, +\infty} \llbracket \cdot \rrbracket.$$

Again, by applying Proposition 3.11 to $K^n \odot \tilde{N}^{(a)}$ and $\widehat{M}^{(a)}$, we conclude that there exists $\Phi^{(a)} \in L_{loc}^2(\widehat{M}^{(a)}, \mathbb{G})$ such that

$$\langle K^n \odot \tilde{N}^{(a)}, \widehat{M}^{(a)} \rangle^{\mathbb{G}} \text{ converges locally in } L^1 \text{ to } \langle \Phi^{(a)} \cdot \widehat{M}^{(a)}, \widehat{M}^{(a)} \rangle^{\mathbb{G}}.$$

Recall that $I_{\tau, +\infty} \llbracket \cdot \langle M, m \rangle^{\mathbb{F}} \rrbracket = \widehat{\Phi}_2 \cdot \langle \widehat{M}^{(a)} \rangle^{\mathbb{G}}$ in (3.58). Thus, the uniqueness of the limit leads to

$$I_{\tau, +\infty} \llbracket \frac{-1}{1 - Z_-} \cdot \langle M, m \rangle^{\mathbb{F}} - \widehat{\lambda} I_{\tau, +\infty} \llbracket \cdot \langle M \rangle^{\mathbb{F}} \rrbracket = \left(\Phi^{(a)} - R^{(a)} \widehat{\Phi}_2 \right) \cdot \langle \widehat{M}^{(a)} \rangle^{\mathbb{G}}. \quad (4.75)$$

Due to the locally boundedness of $(1 - Z_-)^{-1}I_{\mathbb{T}, +\infty}$, it is easy to see that

$$\widehat{\lambda}^{\mathbb{G}} := \left(\Phi^{(a)} - R^{(a)} \widehat{\Phi}_2 \right) \in L_{loc}^2(\widehat{M}^{(a)}, \mathbb{G}), \quad (4.76)$$

and satisfies

$$I_{\mathbb{T}, +\infty} \cdot S = \widehat{M}^{(a)} + \widehat{\lambda}^{\mathbb{G}} \cdot \langle \widehat{M}^{(a)}, \widehat{M}^{(a)} \rangle^{\mathbb{G}}. \quad (4.77)$$

This proves the Structure Conditions for $S - S^\tau$ under \mathbb{G} , and the proof of the theorem is completed. \square

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